Math 105, Assignment 5 **Due 2015-03-30, 4:00 pm**

In the following problems you are expected to justify your answers unless stated otherwise. Answers without any explanation will be given a mark of zero. The assignment needs to be in my hand before I leave the lecture room or you will be given a zero on the assignment! **Don't forget to staple your assignment! You may lose a mark if you do not.**

Recall: Let X be a continuous random variable with PDF $p_X(t)$ and $E(X) = \mu$. For any function f, we say that

$$\begin{aligned} \mathbb{E}(f(X)) &= \int_{-\infty}^{\infty} f(t)p_X(t)dt \\ \text{Var}(X) &= \mathbb{E}[(X-\mu)^2] \\ \sigma(X) &= \sqrt{\text{Var}(X)} \end{aligned}$$

1. For this question refer to the diagram of the spinner on page 9.8 in the notes. Let θ (as in the notes) denote the angle the spinner makes with the positive x-axis, taking values in $[0, 2\pi)$. Our spinner in class was equally likely to point in any direction. However, we spun our spinner so many times, it doesn't spin as well anymore and slows down in some parts of the board. The PDF of θ is

$$p_\theta(t) = \begin{cases} C\sin^2(\frac{t}{2}) & 0 \le t < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Find the following:

(a) C.

(b) The CDF of θ, $F_\theta(t)$

(c) $\mathbb{P}$(spinner lands in the left half the board), $\mathbb{P}(\tan\theta > 1)$

(d) $\mathbb{E}(\theta)$

(e) **Bonus:** Suppose we play a game where you pay me k dollars to spin the spinner, and I give you $100\cos^2(\theta/2)$ dollars in return. What is the highest value of k that you should be willing to pay me?

2. Let X be a continuous random variable with pdf $p_X(t)$, $\mu = \mathbb{E}(X)$. Show that:

$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2.$$

Hint: We have that

$$\text{Var}(X) = \int_{-\infty}^{\infty} (t-\mu)^2 p_X(t)dt.$$

Now expand the right hand side.

3. For each of the following examples find:

- CDF for (a) and PDF for (b)
- $\mathbb{E}(X)$
- $\text{Var}(X)$ **Hint:** Use question 2

(a) X is exponentially distributed with rate λ, ie

$$p_X(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) Y has the cdf

$$F_Y(t) = \begin{cases} 0 & t < 0 \\ t^4 & 0 \leq t \leq 1 \\ 1 & 1 < t \end{cases}$$

4. Let X be a continuous random variable with pdf $p_X(t)$,and $a, b \in \mathbb{R}$. If $Y = aX + b$, show the following:

   (a) $\mathbb{E}(Y) = a\mathbb{E}(X) + b$

   (b) $\text{Var}(Y) = a^2\text{Var}(X)$

   (c) It is 1pm. You are famished. On any other day you would struggle for an hour trying to decide where to spend your hard earned money to silence your stomach. But not today. Today is Tuesday, which means \$3.87 burgers at Triple O's$^{\text{TM}}$. You get there and see the crowd of people in a line leaving the shelter of Triple O's$^{\text{TM}}$ all the way past Sauder. It begins to rain. You are cold, alone, exposed to the elements. It doesn't matter, you came for a cheap burger and dammit you will get one. You wait.

   Let X be the time you spend waiting in line to make your order, X is exponentially distributed with an average waiting time of 30 minutes ($\lambda = 1/30$). Suppose once you make your order, you have to wait an additional 15 minutes to receive your mythical burger with a random limp pickle on top. Suppose further you value your free time at 25 cents per minute (time is money and you aren't cheap). Let Y denote the amount money you spent, both with your credit card and with your life.

      i. Justify that $Y = 3.87 + (0.25)(15) + 0.25X$

      ii. Find $\mathbb{E}(Y)$ (the expected value of your life you expect to spend)

      iii. Find $\sigma(Y)$ (how much you will deviate from the mean on average), rounded to the nearest cent. (

   Hint: Even if you didn't get question 3, you may assume that given X is exponentially distributed with rate λ, $\mathbb{E}(X) = \frac{1}{\lambda}$ and $\text{Var}(X) = \frac{1}{\lambda^2}$ respectively.

   Was it worth it? (The latter question is one you only need to answer to yourself.)